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A(17,6,4) = 20 OR THE NONEXISTENCE OF THE SCARCE DESIGN SD(4,1 ;17,21)

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A(17,6,4) = 20 or the nonexistence of the scarce design SD(4,1;17,21)

by

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ABSTRACT

In this note it is shown that a binary constant weight code with word length n = 17, weight w = 4 and minimum distance d = 6 cannot have 21 code words (which is the value of the Johnson bound). It follows that A(17,6,4) = 20.

KEY WORDS & PHRASES: *packing, constant weight code*

0. INTRODUCTION

In [7] Johnson gave an upper bound for $A(n,d,w)$, the maximum number of code words possible in a binary code consisting of vectors with length n, weight w and minimum mutual distance d. It seems that, given d and w, this upper bound is attained for sufficiently large n. For instance, in BROUWER & SCHRIJVER [2] it is shown that for n sufficiently large and $n \not\equiv 5 \pmod 6$, $A(n,6,4) = J(n,6,4)$, where $J(n,d,w)$ is the Johnson bound. Recall that for $n \geq 1$, $d \geq 1$ and $w \geq 0$ the number $J(n,d,w)$ is defined recursively by

$$J(n,d,w) = 1 \qquad \text{for } w \leq n < d \text{ or } w = 0 \text{ or } w = n,$$

$$J(n,d,w) = 0 \qquad \text{for } w > n,$$

and

$$J(n,d,w) = \min\left\{\left\lfloor\frac{n}{w} J(n-1,d,w-1)\right\rfloor, \left\lfloor\frac{n}{n-w} J(n-1,d,w)\right\rfloor\right\} \text{ otherwise.}$$

In the case of $d = 6$, $w = 4$, we find

$$J(n,6,4) = \begin{cases} \left\lfloor\frac{n}{4}\left\lfloor\frac{n-1}{3}\right\rfloor\right\rfloor & \text{for } n \not\equiv 1 \pmod 3, \\ \left\lfloor\frac{n}{4}\left\lfloor\frac{n-1}{3}\right\rfloor\right\rfloor - 1 & \text{for } n \equiv 1 \pmod 3. \end{cases}$$

For n in one of the residue classes 0,1,2,3,4 mod 12 we have $A(n,6,4) = J(n,6,4)$ without exception; in this note it is shown that this is not true for the residue class 5 mod 12 by proving that

$$A(17,6,4) = J(17,6,4) - 1 = 20.$$

Concerning the other residue classes, it is known that

$$A(8,6,4) = J(8,6,4) - 2 = 2,$$
$$A(9,6,4) = J(9,6,4) - 1 = 3,$$
$$A(10,6,4) = J(10,6,4) - 1 = 5,$$
$$A(11,6,4) = J(11,6,4) - 1 = 6,$$

while no exceptions are known in the residue classes 6 and 7 mod 12. (I conjecture, however, that 18 and 19 are such exceptions.)

We will use a mixture of coding theoretic, set theoretic and graph theoretic language.

If we fix a set X of n elements, then a code word $\underline{u}$ of weight w is identified with the subset U of X of which it is the characteristic function (so that $|U| = w$); it is also identified with the complete subgraph K_w of K_n containing the points of U as vertices.

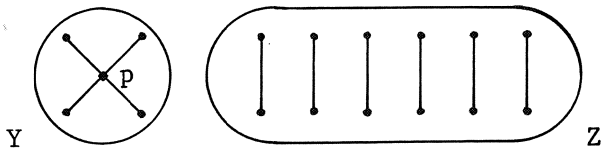
Translated into set theory, our problem becomes one about scarce designs as studied by HANANI [6]; in fact we prove the nonexistence of SD(4,1;17,21).

In terms of graph theory we look for an edge-disjoint packing of K_4's into K_{17} (for: d = 6 means that two 4-tuples have at most one point in common), a type of problem studied by GUY [4], BEINEKE [1], and CHARTRAND, GELLER & HEDETNIEMI [3].

1. THE PROOF

In this section we prove $A(17,6,4) \neq 21$. Suppose $A(17,6,4) = 21$, so that we have a collection $\mathcal{C}$ of 21 4-subsets of a 17-set X such that two of those 4-subsets have at most one point in common.

Let $G = (X,\Gamma)$ be the graph with vertex set X and as edges all pairs not occurring in one of the elements of $\mathcal{C}$. Since K_{17} has $\binom{17}{2} = 136$ edges, while only $21.\binom{4}{2} = 126$ are covered by elements of $\mathcal{C}$, it follows that $|\Gamma| = 10$. In K_{17} each point has valency 16, while removing a K_4 decreases valencies by 0 or 3; hence in G each point x has a valency $v_x \equiv 1 \pmod 3$. Since G has 17 points and only 10 edges it follows that exactly one point has valency 4 and all other points have valency 1. Let p be the point with valency 4. Necessarily, the graph G looks as follows:



Let H be the complementary graph of G, i.e., H is the union of the elements of $\mathcal{C}$.

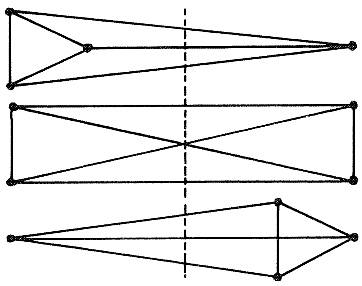
Let Y be the set of 5 points consisting of p and its neighbours, and let $Z = X \setminus Y$.

Now consider the 4-tuples $U \in \mathcal{C}$ that contain an edge between Y and Z. There are three types:

(a) $|U \cap Y| = 3$;

(b) $|U \cap Y| = 2$;

(c) $|U \cap Y| = 1$.



The complete graph K_Z on the points of Z has $\binom{12}{2} = 66$ edges of which 6 are in G, so that 60 must be covered by sets $U \in \mathcal{C}$. But the number of edges between Y and Z is also 60 (=5.12). Therefore the first possibility one could try in order to construct the required packing is to use only sets of type (c), namely 20 of them, while the last 4-tuple is used to complete $G \cap Y$ to a K_5. Let us eliminate this case first.

A. Assume $\mathcal{C}$ contains a set U_0 contained in Y so that $(G \cap Y) \cup U_0 = K_Y$. Since all edges within Y have been used, all sets intersecting both Y and Z must be of type (c), and by the above counting argument there are no sets in $\mathcal{C}$ contained entirely within Z.

Let r be a new point and consider the collection of triples

$$\mathcal{D} = \{U \cap Z \mid U \in \mathcal{C} \setminus \{U_0\}\} \cup \{\{z_1, z_2, r\} \mid (z_1, z_2) \in \Gamma\}.$$

Since $\mathcal{D}$ covers each pair in $Z \cup \{r\}$ exactly once, it is a Steiner triple

system on these 13 points.
Next, if $y \in Y$, then the four triples $\{U \cap Z \mid y \in U \in C\setminus\{U_0\}\}$ are disjoint, so we have here a Steiner triple system on 13 points with the special property that it contains five groups of four pairwise disjoint triples, none of them containing a fixed point r.
Now HALL [5,p.237] states that there exist exactly two nonisomorphic Steiner triple systems on 13 points, and gives them explicitly.
The first one is essentially the triple system obtained by adding all residues mod 13 to the blocks {1,2,5} and {1,7,9}. For each point r it contains exactly one group of four pairwise disjoint triples not containing r (e.g. for r = 0 we get the triples {7,8,11}, {1,6,12}, {9,2,4} and {3,5,10}). [REMARK : its automorphism group is generated by

$$\pi_1 = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)$$

and

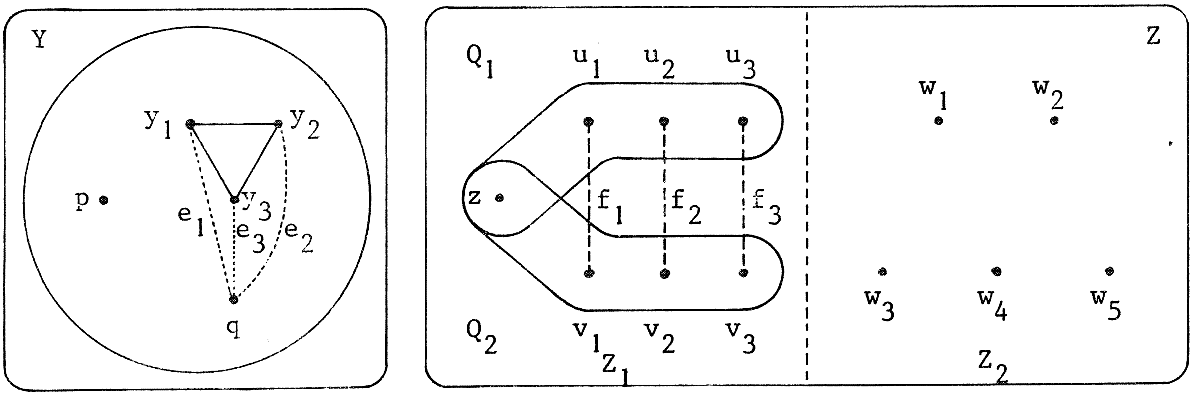
$$\pi_2 = (0)\ (1\ 9\ 3)\ (6\ 2\ 5)\ (4\ 10\ 12)\ (11\ 8\ 7);$$

the 26 triples can be divided into two classes: those corresponding to {1,2,5} occur in exactly one group of four distinct triples and those corresponding to {1,7,9} occur in three such groups each.]
The other one can be obtained from the first one by replacing the four triples {9,11,3}, {8,1,3}, {9,1,7}, {8,11,7} by {9,11,7}, {8,11,3}, {8,1,7} and {9,1,3}. It has less symmetry [its automorphism group being generated by π_2] and contains four points (namely 1,3,9 and 0), each having two groups of four disjoint triples in their complement, while the other nine points do not have a group of four disjoint triples in their complement.
In both cases we never find five groups, so this case is finished.

Since we have excluded the case in which the six edges in Y are covered by one 4-tuple contained in Y, they must be covered by 4-tuples of type (a) or (b): either 2 of type (a) or 1 of type (a) and 3 of type (b) or 6 of type (b). This gives us the cases B, C and D respectively.

B. Two 4-tuples of type (a). This is impossible since the K_4 $Y\setminus\{p\}$ is not the union of two copies of K_3.

C. One 4-tuple of type (a) and three 4-tuples of type (b). In order to cover the 60 edges between Y and Z we need 15 4-tuples of type (c), which leaves 60 − (3 + 45) = 12 edges within Z; that is, there are two 4-tuples contained entirely within Z, say Q_1 and Q_2. Let the 4-tuple of type (a) be $\{z,y_1,y_2,y_3\}$ with $z \in Z$ and let $Y = \{p,q,y_1,y_2,y_3\}$. Now consider the point z. It needs to have 10 edges within Z (i.e. has valency 10 in $H \cap Z$), while 4-tuples meeting both Y and Z can provide at most 4 (namely using two sets of type (c) with vertices p and q); therefore z must be in both sets Q_1 and Q_2.



Each point in Z is incident with ten edges in the graph $H \cap Z$, which is an even number; also, each time a set of type (c) is used an even number is removed; therefore the union of the graphs Q_1, Q_2 and those of type (b) has an even degree at each point of Z. Since $Q_1 \cup Q_2$ has six points of odd degree this means that the three edges f_1, f_2 and f_3, which are the intersections of the sets U_1, U_2 and U_3 of type (b) with Z, all intersect both Q_1 and Q_2. Hence $Q_1 = \{z,u_1,u_2,u_3\}$, $Q_2 = \{z,v_1,v_2,v_3\}$, $U_i = \{q,y_i,u_i,v_i\}$ $(i = 1,2,3)$.

This gives us 6 + 6 + 3 = 15 edges on the points of $Q_1 \cup Q_2$. But $\binom{7}{2} = 21$, so six are missing. Three of them could be edges of G, but at least three must be edges of 4-tuples of type (c). However, such a 4-tuple intersects both Q_1 and Q_2 in at most one point, and hence we need three separate 4-tuples.

Let $Z_1 = Q_1 \cup Q_2$ and $Z_2 = Z \backslash Z_1 = \{w_1,w_2,w_3,w_4,w_5\}$.

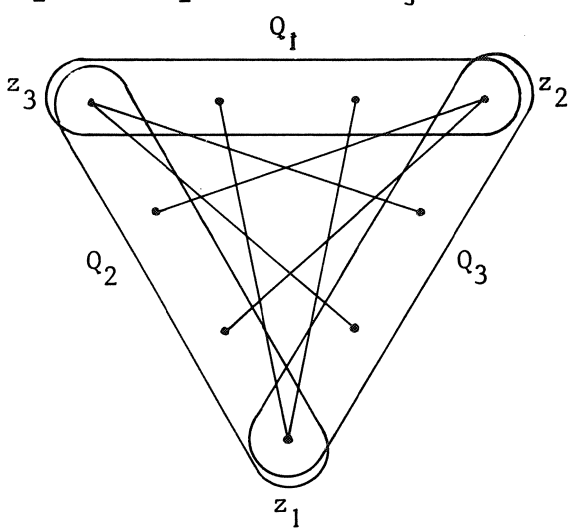
Now consider the point p. It is in four sets of type (c) and defines four triangles within Z. But for such a triangle at most one edge can be

contained in Z_1. On the other hand, if one triangle is contained entirely within Z_2, then of the remaining three at least one must be contained entirely within Z_1, which is impossible. Hence by counting it follows that three triangles intersect Z_1 in two points while the last one intersects Z_1 in one point, which then has to be the point z. Therefore without loss of generality we may assume that the four triangles are $\{z,w_1,w_2\}$, $\{u_1,v_2,w_3\}$, $\{u_2,v_3,w_4\}$ and $\{u_3,v_1,w_5\}$.
Next consider the point q. It also induces a partition of Z, namely into the three pairs $f_i = \{u_i,v_i\}$ $(i = 1,2,3)$ and two triangles covering $\{z,w_1,w_2,w_3,w_4,w_5\}$. But the triangle containing z cannot contain either w_1 or w_2. Hence both must be in the other triangle, which means that the edge $\{w_1,w_2\}$ is covered twice, a contradiction. This completes case C.

D. No sets of type (a) and six 4-tuples of type (b). In order to cover the 60 edges between Y and Z we need 12 sets of type (c), which leaves $60 - (6+36) = 18$ edges within Z. That is, there are three 4-tuples contained entirely within Z, say Q_1, Q_2 and Q_3. Now we get five subcases according to the relative position of Q_1, Q_2 and Q_3.

D_1. $Q_1 \cap Q_2 \cap Q_3 = \{z\}$. This is impossible, since now the point z needs connection with five points in Y while only two edges are left within Z; but two sets of type (b) cover at most four points in Y. (In fact only one edge is left within Z since the other is in G.)

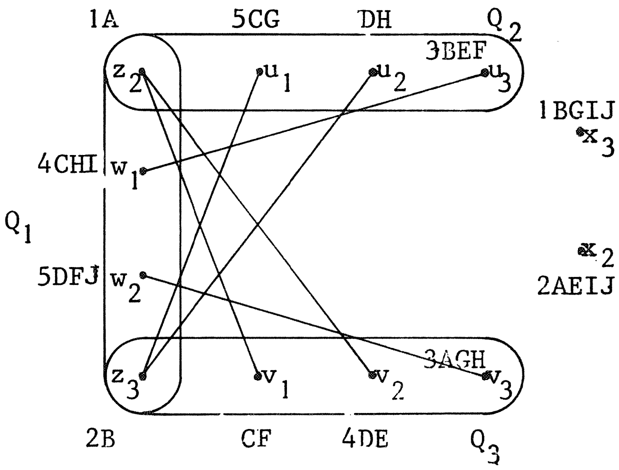
D_2. Each Q_i intersects Q_j $(1\leq i,j\leq 3)$, but $Q_1 \cap Q_2 \cap Q_3 = \emptyset$. Say



$Q_1 = \{z_2,z_3,u_1,v_1\}$,
$Q_2 = \{z_1,z_3,u_2,v_2\}$,
$Q_3 = \{z_1,z_2,u_3,v_3\}$.

After removal of the edges of G and Q_1, Q_2 and Q_3 from K_z, the points u_i and v_i have odd valency, i.e. each of them must be in a set of type (b). Each of the points z_i needs connection with five points in Y while only $5 - 1 = 4$ edges are left within Z; consequently, the only possibility is that each of the points z_i is in two sets of type (b) and in one of type (c) (this one containing p). But this means that all sets of type (b) have their points in Z in $Q_1 \cup Q_2 \cup Q_3$. Now consider the point p; it induces a partition of Z into four triangles; but such a triangle can have at most one point in common with $Q_1 \cup Q_2 \cup Q_3$, which is impossible.

D_3. Q_1 intersects both Q_2 and Q_3, but $Q_2 \cap Q_3 = \emptyset$. Let



$$Q_1 = \{z_2, z_3, w_1, w_2\},$$
$$Q_2 = \{z_2, z_1, u_2, u_3\},$$
$$Q_3 = \{z_3, v_1, v_2, v_3\},$$

and

$$Z \setminus Q_1 \cup Q_2 \cup Q_3 = \{x_3, x_2\}.$$

Consider the point p; it induces a partition of Z into four triangles, and we may safely assume that these are

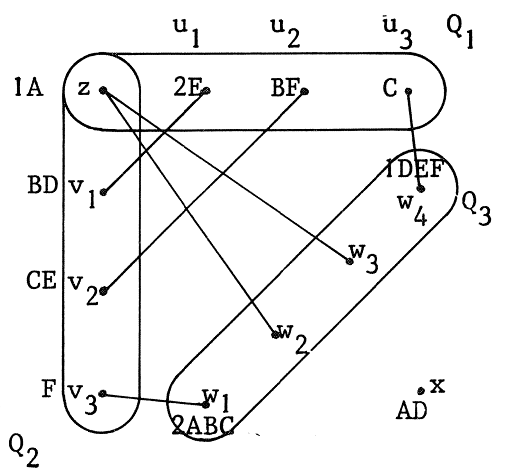
$$A: \{z_2, x_2, v_3\},\ B: \{z_3, x_3, u_3\},\ C: \{u_1, v_1, w_1\},\ D: \{u_2, v_2, w_2\}.$$

(Here the labels are used to mark the points in the figure; note that two points cannot have a common pair of labels.)

Next look at the 6 edges stemming from the sets of type (b): by the same reasoning as before, the points z_2 and z_3 must each be incident with two of them, while $u_1, u_2, u_3, v_1, v_2, v_3, w_1$ and w_2 are incident with one each. Therefore these 6 edges are $\{z_2, v_1\}$, $\{z_2, v_2\}$, $\{z_3, u_1\}$, $\{z_3, u_2\}$, and $\{u_3, w_1\}$, $\{v_3, w_2\}$, since w_1 and w_2 were still symmetric.

Now look at the 6 edges of the graph $G \cap Z$: we must have 1: $\{z_2,x_3\}$ and 2: $\{z_3,x_2\}$. Also, 3: $\{u_3,v_3\}$ follows, since v_3 is connected to each of the points $\{w_2,v_1,v_2,v_3,x_2\}$ not connected with u_3. The remaining points $\{w_2,v_1,v_2,x_2\}$ are covered by two triangles with top u_3 which necessarily are E: $\{v_2,x_2,u_3\}$ and F: $\{w_2,v_1,u_3\}$. Likewise we find the triangles G: $\{u_1,x_3,v_3\}$ and H: $\{w_1,u_2,v_3\}$ with top v_3. The point w_1 is not yet connected with $\{v_2,x_2,x_3\}$. One connection must be an edge of G and the other a triangle; but we know already the edges of G containing x_2 and x_3, and hence we have 4: $\{w_1,v_2\}$ and I: $\{x_2,x_3,w_1\}$. Likewise we find 5: $\{w_2,u_1\}$ and J: $\{x_2,x_3,w_2\}$. But now the edge $\{x_2,x_3\}$ has been covered twice, a contradiction.

D_4. Q_1 intersects Q_2, but $Q_3 \cap (Q_1 \cup Q_2) = \emptyset$. Let



$$Q_1 = \{z,u_1,u_2,u_3\},$$
$$Q_2 = \{z,v_1,v_2,v_3\},$$
$$Q_3 = \{w_1,w_2,w_3,w_4\}$$

and

$$Z \setminus Q_1 \cup Q_2 \cup Q_3 = \{x\}.$$

Looking at the edges stemming from sets of type (b) we may suppose that they are $\{z,w_2\}$, $\{z,w_3\}$, $\{w_1,v_3\}$, $\{w_4,u_3\}$, $\{u_1,v_1\}$ and $\{u_2,v_2\}$.
The point z is in one triangle, which we may assume to be A: $\{z,w_1,x\}$.
The edge of G containing z is then 1: $\{z,w_4\}$.
The point w_1 is not yet connected with the points in $\{v_1,v_2,u_1,u_2,u_3\}$.
Neither of the edges $\{v_1,w_1\}$ and $\{v_2,w_1\}$ can be in G, since otherwise the four remaining points cannot be covered by two triangles with top w_1.
We may therefore assume the existence of the triangle B: $\{w_1,v_1,u_2\}$, and either both $\{w_1,v_2,u_1\}$ and $\{w_1,u_3\}$ or both $\{w_1,v_2,u_3\}$ and $\{w_1,u_1\}$. In the first case there are no edges available for three triangles with top w_4, and hence we draw C: $\{w_1,v_2,u_3\}$ and 2: $\{w_1,u_1\}$. The three triangles containing w_4 each contain one point from $\{v_1,v_2,v_3\}$; but the one containing v_1 cannot contain either u_1 or u_2, and hence it is D: $\{w_4,v_1,x\}$.

Now the other two have to be E: $\{w_4,v_2,u_1\}$ and F: $\{w_4,v_3,u_2\}$.
Next consider the points not yet connected with w_2: $\{x,u_1,u_2,u_3,v_1,v_2,v_3\}$. Both u_2 and v_2 are already connected with each point of $\{u_1,u_2,u_3,v_1,v_2,v_3\}$, and hence one of them is connected to w_2 by an edge of G while the other is in a triangle together with w_2 and x. This leaves the points $\{u_1,u_3,v_1,v_3\}$ which necessarily are covered by the triangles $\{w_2,u_1,v_3\}$ and $\{w_2,u_3,v_1\}$. Unfortunately, however, by exactly the same reasoning we conclude the existence of the triangles $\{w_3,u_1,v_3\}$ and $\{w_3,u_3,v_1\}$, a contradiction.

D_5. The sets Q_1, Q_2 and Q_3 are pairwise disjoint. This time a study of the covering of Z does not lead to a contradiction, since it is quite possible to make a B({2,3,4},1,12) design of the required type on the points of Z. The difficulty lies in the way the triangles are connected with points of Y.
First we observe that this time each point of Z lies in exactly one Q_i and in exactly one set of type (b). Returning to the coding point of view we write down the six code words corresponding to sets of type (b):

p	Y\{p}	Z
0	1100	11 00 00 00 00 00
0	1010	00 11 00 00 00 00
0	1001	00 00 11 00 00 00
0	0110	00 00 00 11 00 00
0	0101	00 00 00 00 11 00
0	0011	00 00 00 00 00 11

Next we write down the eight code words corresponding to sets of type (c) not containing p:

0	1000	00 00 00 10 10 10
0	1000	00 00 00 01 01 01
0	0100	00 10 10 00 00 10
0	0100	00 01 01 00 00 01

We can reach this far without restriction of generality, but for the next two lines there are two possibilities:

D_5(i):

0	0010	10 00 10 00 01 00
0	0010	01 00 01 00 10 00

This is the easy case. We can rapidly see that it is now impossible to add the three vectors corresponding to Q_1, Q_2 and Q_3: try to fill three rows simultaneously such that in each of the last 12 positions exactly one of them contains a one, and such that the inner product with each of the previous rows is at most one. We get

$$\begin{array}{l} 00\ 00\ 10\ 00\ \underline{1}0\ 0\underline{1} \\ ..\ 0.\ 0.\ 01\ 00\ \underline{1}0 \\ 0.\ ..\ 0.\ 10\ 01\ 00 \end{array}$$

or

$$\begin{array}{l} 01\ 10\ 00\ 10\ \underline{0}0\ 0\underline{1} \\ 00\ 00\ 01\ 00\ 01\ \underline{1}0 \\ 10\ 00\ 10\ 01\ 10\ 00 \end{array}$$

where the underlined digits are the assumptions, and the rest follows.

The other case goes as follows:

D_5(ii):

0	0010	10 00 10 00 10 00
0	0010	01 00 01 00 01 00
0	0001	10 .. 00 .. 00 00
0	0001	01 .. 00 .. 00 00

Again, trying to fill three rows of weight four, we get

```
10 .0 00 .0 00 01
.. 0. 0. 0. 0. 10
.. .. 1. .. 0. 00
```

or

```
0. .0 10 .0 00 01
.. 0. 0. 0. 0. 10
1. .. 0. .. 0. 00
```

or

```
01 00 10 10 00 01
10 01 00 00 01 10
00 10 01 01 10 00
```

or

```
01 10 00 00 10 01
10 00 01 01 00 10
00 01 10 10 01 00
```

Unfortunately the last two cases do not yet yield a contradiction, but they both imply that the two rows which were left open are

```
0    0001    10 10 00 10 00 00
0    0001    01 01 00 01 00 00
```

However, as soon as we try to add 4 triples stemming from a four-tuple containing p we get:

$$\begin{matrix} 01 & 00 & 10 & 10 & 00 & 01 \\ 10 & 01 & 00 & 00 & 01 & 10 \\ 00 & 10 & 01 & 01 & 10 & 00 \\ \underline{1}0 & 00 & 00 & 00 & 00 & 01 \end{matrix}$$

or

$$\begin{matrix} 01 & 10 & 00 & 00 & 10 & 01 \\ 10 & 00 & 01 & 01 & 00 & 10 \\ 00 & 01 & 10 & 10 & 01 & 00 \\ \underline{1}0 & 01 & 00 & 00 & 00 & 00 \end{matrix}$$

a contradiction both times.

This completes the proof. □

2. A(17,6,4) = 20

As explained in the introduction we have by the Johnson bound

$$A(17,6,4) \leq J(17,6,4) = \left\lfloor \frac{17}{4} \cdot \left\lfloor \frac{16}{3} \right\rfloor \right\rfloor = 21.$$

Examining all possible cases we found in the previous section that $A(17,6,4) \neq 21$ so that $A(17,6,4) \leq 20$. But it is easy to exhibit 20 words of weight 4 and with word length 17 and at minimal mutual distance 6; in fact, we even have $A(16,6,4) \geq 20$, as is shown by the affine plane AG(2,4).

Therefore $A(16,6,4) = A(17,6,4) = 20$.

REFERENCES

[1] BEINEKE, L.W., *On the decomposition of complete graphs into planar and other subgraphs*, Doctoral thesis, Univ. of Michigan (1965).

[2] BROUWER, A.E. & A. SCHRIJVER, *An infinite sequence of optimal constant weight codes, or the packing of cocktail party graphs with* K_4*'s*, Report ZW 64 (Mathematisch Centrum, Amsterdam, 1975).

[3] CHARTRAND, G., D. GELLER & S. HEDETNIEMI, *Graphs with forbidden subgraphs*, JCT 10 (1971) 12 - 41.

[4] GUY, R.K., *A problem of Zarankiewicz*, Research paper no.12, Univ. of Calgary (1967).

[5] HALL Jr., M., *Combinatorial Theory* (Blaisdell, Waltham, Mass., 1960).

[6] HANANI, H., *Balanced incomplete block designs and related designs*, Discrete Math. 11 (1975) 255 - 370.

[7] JOHNSON, S.M., *A new upper bound for error-correcting codes*, IRE Trans. Inf. Theory IT - 8 (1962) 203 - 207.